Week 4- Lecture 2

Computational Methods II (Elliptic)

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G for Errors using Point Jacobi (cont'd)

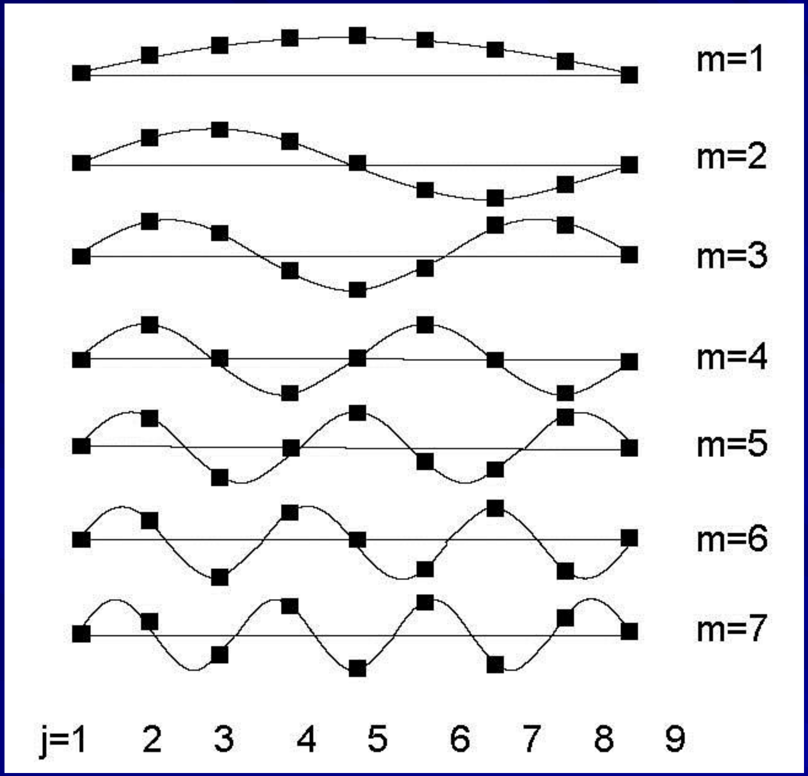
- The plot shows how the errors decay for various error modes (in terms of sines and cosines)

- Consider 1D situation – let there be M+1 points on a line giving M intervals. The errors can be expressed as

$$f_m(j) = sin\frac{m\pi j}{M}$$

where m = 1,2, .., M-1

Error Modes for M=8



Seven possible error modes on a 1D mesh with eight intervals –
Lowest to highest frequencies.

Error Modes Analysis M=8

- Any numbers satisfying f(0)=f(M)=0 can be represented as

$$u_j = \sum_{m=1}^{m=M-1} \left[c_m f_m(j) \right]$$

- This is a discrete Fourier Transform

- In 1D, need only to consider the following set of discrete frequencies (lowest to highest)

$$\theta = \frac{\pi}{M}, \frac{2\pi}{M}, \ldots, \frac{(M-1)\pi}{M}$$

2D situation

- Consider 2D situation – let there be M+1 and N+1 points, giving M and N intervals in x and y directions

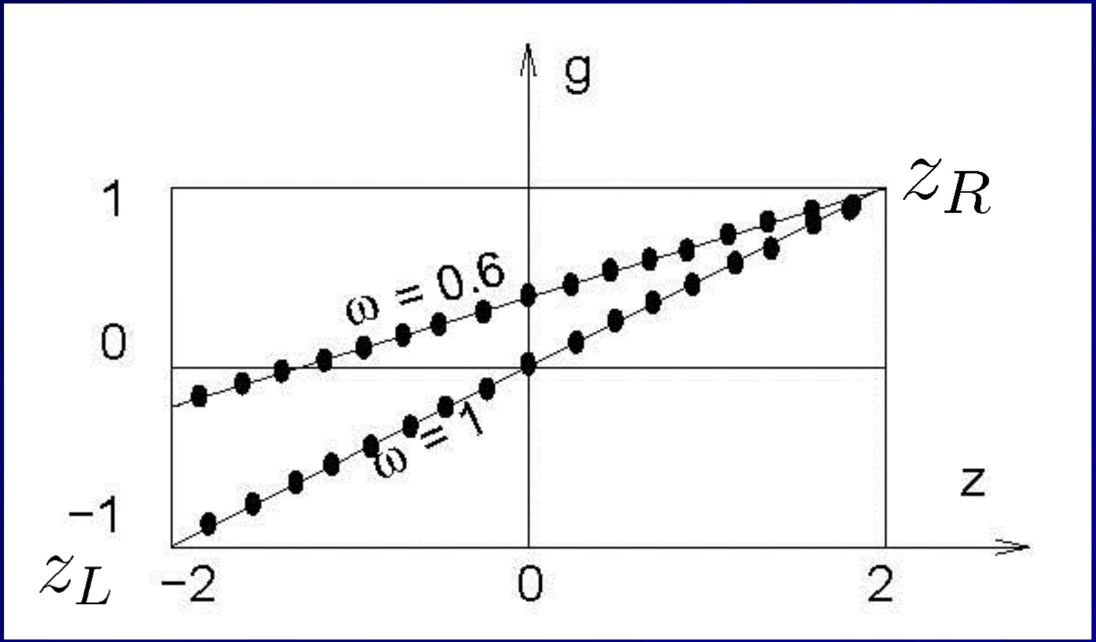
$$f_m(j,k) = sin\frac{m\pi j}{M} sin\frac{n\pi k}{N}$$

where m = 1,2, .., M-1. n=1,2,.. N-1.

Assume M=N, the highest and lowest frequencies correspond to

$$z_l = cos\frac{(M-1)\pi}{M}, z_r = cos\frac{\pi}{M}$$

Discrete G versus z



- Each point corresponds to discrete error modes, each error mode decays differently even for same relaxation factor

- Note $Z_L = -Z_R$

Decay of Error Modes

- In eliminating the errors, we do not care the sign of amplitude, just the magnitude of it – want g as small as possible

- If $0 < |g| < 1$ for ALL error modes, then we could say that we eventually will remove all the iteration errors and hence a steady state solution will be obtained.

- However, the errors will be removed at different rates, with modes corresponding to the largest $|g|$ (convergence rate) will be most difficult to be removed- dominant error mode

Decay of Error Modes (cont'd)

- For ω=0.6, it is clear that Z_R (lowest frequency) is the dominant error mode

$$\begin{aligned} g &= 1 - \frac{\omega}{2}(2 - z_R) \\ &= 1 - \frac{\omega}{2}(2 - cos\frac{\pi}{M}) \end{aligned}$$

- Increasing ω, will decrease g until ω=1 – both lowest and highest frequency modes become dominant

- Beyond $\omega > 1$, Z_L (highest frequency) modes become more dominant and |g| deteriorates.

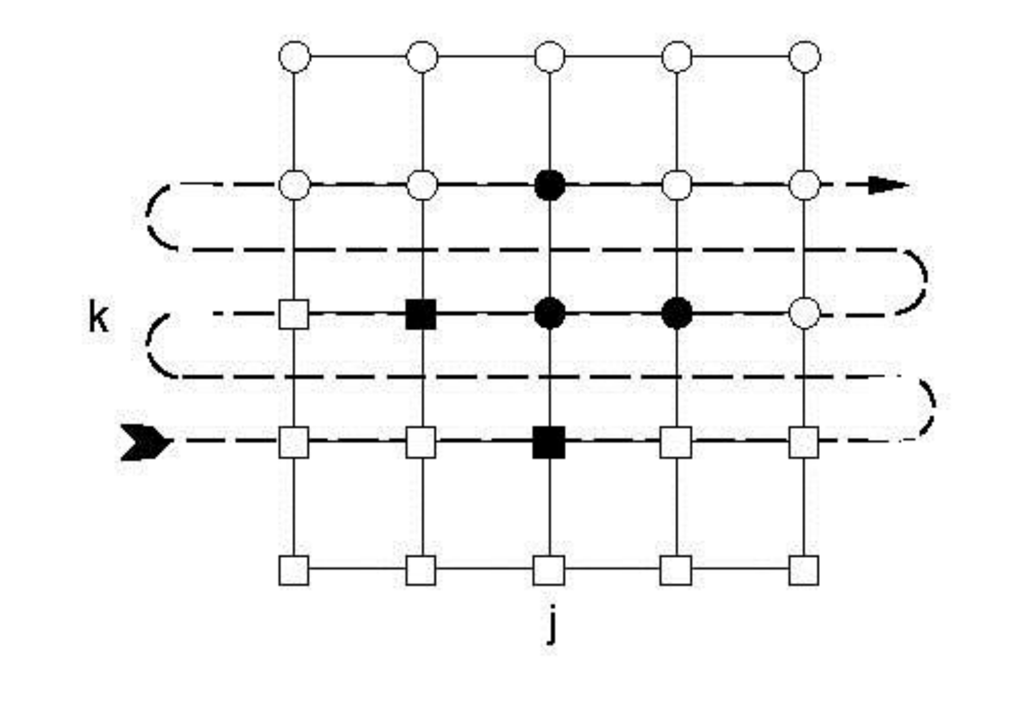
Computational Cost

- Recall for direct method (Gaussian Elimination) requires $O(M^6)$ in 2D and $O(M^9)$ in 3D.

- Using point Jacobi iteration methods costs $O(M^4)$ in 2D and $O(M^5)$ in 3D.

- Can this be improved? Let say by one order magnitude?

Gauss Seidel



-When visiting node (j,k) at time level n+1, can use information of nodes (j,k-1) at time level n and (j-1, k) at n+1 (square nodes)

- This is Gauss-Seidel method

SOR Method

- Coupling Gauss-Seidel with point Jacobi gives Successive Over- Relaxation (SOR) method

$$u_{j,k}^{n+1} = u_{j,k}^{n} + \omega[\frac{1}{4}(u_{j-1,k}^{n+1} + u_{j+1,k}^{n} + u_{j,k-1}^{n+1} + u_{j,k+1}^{n}) - u_{j,k}^{n}]$$

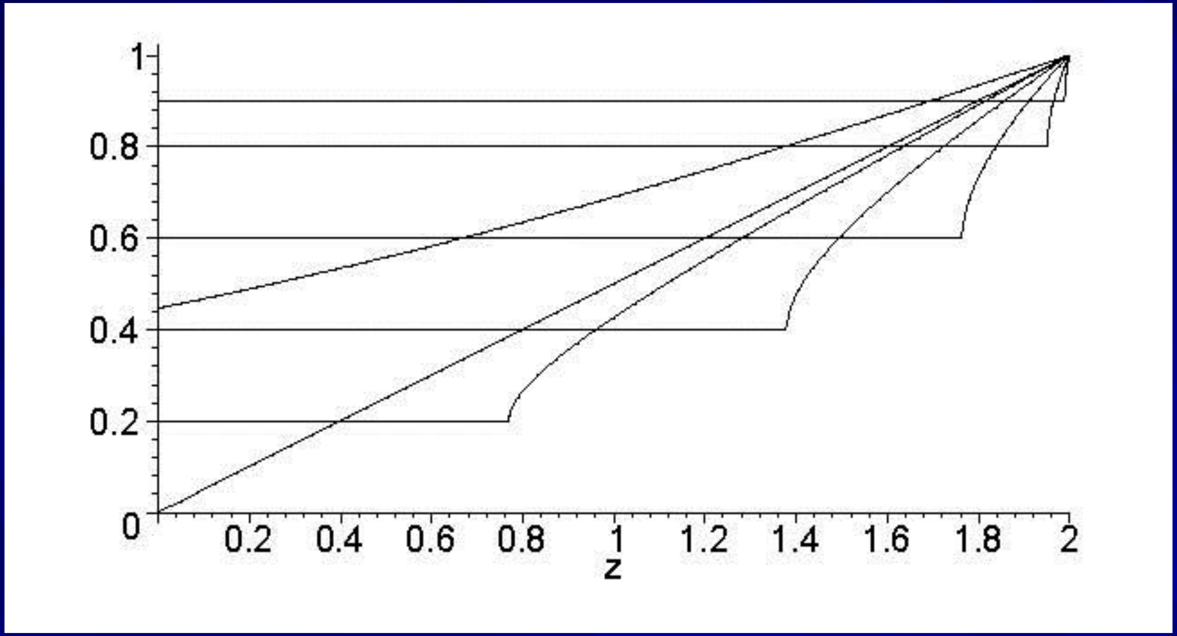
- It can be shown that using VA, the convergence rate

$$g = \frac{\omega z}{4} \pm \sqrt{1 - \omega + \frac{\omega^2 z^2}{16}}$$

- Only consider real z such that

$$z^2 > \frac{16(\omega - 1)}{\omega^2}$$

G versus z for SOR



- Note now z = [0,2]

-Similar to Point Jacobi for ω less than or equal to 1

-But improved convergence rate for ω > 1 (1.16, 1.36, 1.64, 1.81) -> STILL LOWEST FREQUENCY MODE IS DOMINANT

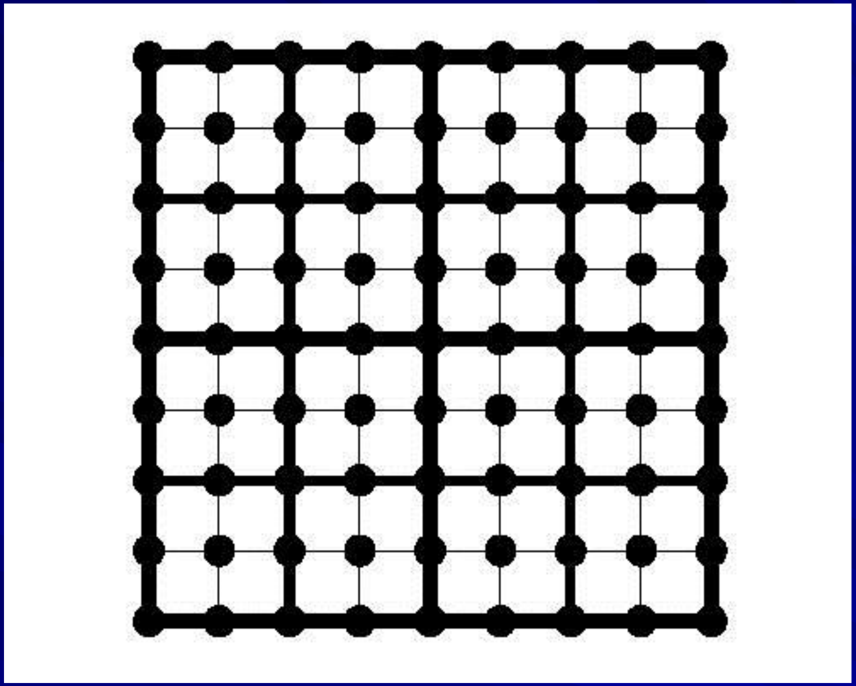
Discussion of Low Frequency Modes

- We have seen that the low frequency modes are most difficult to get rid of for Point Jacobi and SOR methods
- Moderate to high frequency modes are usually much easier to be removed.
- This is the general behavior for almost all methods in solving elliptic PDE
- Need to think out of the box

Multigrid (Brandt)

- Use a hierachy of different grid sizes to solve the elliptic PDE –splitting the error modes
- The low frequency error modes on a fine grid will appear as high frequency modes on a coarser grid.
- Hence easier to be removed
- But there is a problem with coarse grids.

Multigrid Mesh



-An 8 x 8 grid superimposed with 4 x 4 + 2 x 2 grid

Multigrid (cont'd)

- Although coarse grids will remove the error modes quickly, but it will converge to an inaccurate solution
- Need to combine the accuracy of fine grids with the fast convergence on coarse grids
- As long as there are significant error modes in fine grid, coarse grid must be kept to work
- Fine grids must 'tell' the coarse grids the dominant error modes and ask 'advice' on how to remove them – *restriction operator*

Multigrid (cont'd)

- Coarse grids will give information on how to remove the dominant error modes to fine grids and ask for 'advice' regarding accuracy- *prolongation operator*

- Communication between coarse-fine grids is the key

- In between, perform relaxation (solving the steady state equation)

Story

FINE GRID: Excuse me! You upstairs! I'm trying to solve a Poisson equation! So far I've got these residuals. How do you think I should change the solution?

COARSE GRID: Hm! At first glance I think you should make these changes, but before you do, let me ask my upstairs neighbour, who sees a broader picture than me. Excuse me! You upstairs!. . . .

COARSE GRID: Hi! You below! Based on what I can see, and on what I've been told, I think you should make these changes.

FINE GRID: Thanks! Now let me see, if I make those changes what residual do I now find? Well, those are the changes I should recommend to my neighbour below. Hi! You below!. . . .

Multigrid V Cycle