Stability of ODE

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Coding in the machines were the easy part, needed to establish mathematical principles upon which the coding is based.

One of the most important principles is the concept of stability.

Can be illustrated by solving the simple ODE problem
Find $u(t)$ such that

$$\frac{du}{dt} = au$$

$a > 0$, subject to the initial condition (I.C)

$$u(0) = u^0$$

This problem is almost trivial since we know the exact analytical solution.
The analytical solution is

\[ u(t) = u^0 \exp(at) \]

The question is what is the numerical solution?

Assume we are interested in \(0<t<T\), we divide time into small intervals of \(\Delta t\) or

\[ n = \frac{T}{\Delta t} \]

discrete time levels
It is the nth value of u, not u-to-power-n
Use Taylor series to find

\[ u^{n+1} = u^n + \Delta tu^n + O(\Delta t^2) \]

\[ \approx u^n (1 + a\Delta t) \]

This is an example of a numerical method to solve

\[ \frac{du}{dt} = au \]

More generally,

\[ u^{n+1} \approx u^n (1 + \alpha \Delta t^p) \]

**Will this numerical method work?**
It can be shown that the predicted result would be

\[ u(T) = u^0 \left( 1 + \alpha \Delta t^p \right)^{T/\Delta t} \]

Rearrange, see what happens if we make time steps smaller,

\[ u(T) = u^0 \left( 1 + \alpha T \Delta t^{p-1} \frac{\Delta t}{T} \right)^{T/\Delta t} \]

Recall that, hence,

\[ \lim_{n \to \infty} \left( 1 + \frac{a}{n} \right)^n = \exp(a) \]

\[ \lim_{\Delta t \to 0} u(T) = u^0 \exp[\alpha T (\Delta t^{p-1})] \]
As we decrease $\Delta t$ (refinement),

- What happen if $p < 1$ and $\alpha > 0$ ?
- What happen if $p > 1$ and $\alpha > 0$ ?
- What is the correct solution?
If we choose \( p = 1 \)

\[
\lim_{{\Delta t \to 0}} u(T) = u^0 \exp[\alpha T]
\]

Solution is stable, but is it the correct solution? The solution is correct (consistent) if and only if \( \alpha = a \)

For a certain class of linear problems, the Lax Equivalence Theorem says

Stability + Consistency = Convergence

\[
(p = 1) + (\alpha = a) = (\lim_{{\Delta t \to 0}} u(T) \to \exp[\alpha T])
\]
Why would someone be so stupid to choose other than \( p=1 \) and \( \alpha=a \)?

- In more complicated cases, something like this might happen due to either a misconception or

- Programming error!

- The computer does not know what is wrong or right!

- It only knows to process/compute whatever is being fed in!

- **IT IS THE USER’S JOB!**
Stability versus Consistency

- A numerical scheme is unconditionally **stable** if its solution does not ‘blow up’ as the time-steps (or grids) are continuously being refined.

- A numerical scheme is consistent if the *local* difference between the exact solution and numerical solution approaches zero as the time steps (or grids) are refined ($h \rightarrow 0$).